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On Phase Diagram of Ising Model on the Cayley Tree with Restricted Competing Interactions up to the Third-Nearest-Neighbor Generation

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ABSTRACT

We study the phase diagram of the Ising model on Cayley tree with competing the first-, second-, and third-nearest-neighbor interactions. We generalize the approach used by Vannimenus for the Ising model with only first- and second- nearest neignbor interactions. The extension of the anisotropic next-nearest-neighbor Ising (ANNNI) model to include third-nearest-neighbor interactions (A3NNI model) has been proposed by Yamada and Hamaya in an attempt to explain the phase diagram structure of several ferroelectric systems of the type A2BX4. We show that the inclusion of the third-nearest-neighbor interactions is essential for the presence of a periodic +++- - - antiphase <3>. In addition, we discuss similarity, in comparison with the phase diagram of the Vannimenus's approach with competing first- and second-nearest-neighbors. Finally, in the modulated phase was studied in detail where narrow commensurate steps between incommensurate regions appeared when investigating the Lyapunov exponent associated with trajectory of the system.

Keywords: Ising Model, Cayley Tree, Nearest-Neighbor Interactions, Phase Diagram, Modulated Phases.

1. Introduction

Ising models on the Cayley tree with competing interactions has recently been considered extensively because of the appearance of nontrivial magnetic orderings (see Vannimenus (1981),Mariz et al. (1985),Inawashiro and Thompson (1983),Inawashiro et al. (1983),Yokoi et al. (1985),da Silva and Coutinho (1986) and references therein). The Cayley tree is not a realistic lattice; however its amazing topology makes the exact calculation of various quantities possible. For many problems, the solution on a tree is much simpler than it is on a regular lattice and is equivalent to the standard Bethe-Peierls theory Katsura and Takizawa (1974).

More complicated models are studied on tree-like lattices, with the hope to discover new phases. The important point is that statistical mechanics on trees involve nonlinear recursion equations and are naturally connected to the rich world of dynamical systems (see references in Vannimenus (1981)). In addition, the Ising model has found some applications physical, chemical and biological systems, and even in sociology (Kindermann and Snell (1980),Weidlich (1971),Baxter (1982)). In the literature, Ising model on the Cayley tree with competing interactions up to second-nearest-neighbor generation have been studied broadly (Vannimenus (1981),Mariz et al. (1985),Inawashiro and Thompson (1983),Inawashiro et al. (1983),Yokoi et al. (1985),da Silva and Coutinho (1986)). Vannimenus (1981) was considered the case with competing first- and prolonged second-nearest-neighbor interactions, which Vannimenus able to find new modulated phases, in addition to the expected paramagnetic and ferromagnetic phase. Later, Mariz et al. (1985) extended this result, assuming existence on one-level second nearest-neighbor interactions.

In this paper, we study phase diagram of Ising model on the Cayley tree with competing interactions up to third-nearest-neighbor generation (see definitions). Previously, da Silva and Coutinho (1986) studied the Ising model on a Cayley tree of arbitrary order with competing interactions between the first-, second- and third-nearest-neighbor interactions spins belonging to the same branch, and in the presence of an external magnetic field. They have used the approach suggested by by Thompson (1982) for the case of Ising model on Cayley tree with only nearest-neighbor interactions and an external magnetic field.

The authors justified that the inclusion of the third-nearest-neighbor competing interaction is essential for the presence of stable modulated phases. Thus, the aim of this paper to study the phase diagram of the same model considered by da Silva and Coutinho (1986), using approach that suggested by

Vannimenus (1981). A phase diagram of the model describes the morphology of the phases, the stability of phases, the transitions from one phase to another and corresponding transition lines. Here, we use the computational techniques by iterating the recurrence system of equations.

This paper is organized in the following way. The model Hamiltonian is discussed. After that, the recursion equations and the average magnetization are defined. Next section is devoted to a discussion of the phase diagram features. Then, the variations of the wavevector and the Lyapunov exponent for the considered model are discussed detail and the conclusions are given in the last section.

2. Definitions

Cayley tree. A Cayley tree Γ^2 is an infinite tree, a graph without cycles with exactly 3 edges issuing from each vertex. Let denote the Cayley tree as $\Gamma^2 = (V, \Lambda)$, where V is the set of vertices of Γ^2, Λ is the set of edges of Γ^2. Two vertices x and y, $x, y \in V$ are called *first-nearest-neighbor* if there exists an edge $l \in \Lambda$ connecting them, which is denoted by $l =< x, y >$. The distance $d(x, y), x, y \in V$ on the Cayley tree Γ^2, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x, x^0) = n\}, V_n = \{x \in V | d(x, x^0) \leq n\},$$

and L_n denotes the set of edges in V_n. The fixed vertex x^0 is called the 0-th level and the vertices in W_n are called the n-th level. For the sake of simplicity we put $|x| = d(x, x^0), x \in V$.

Two vertices $x, y \in V$ are called *second-nearest-neighbor* if $d(x, y) = 2$. The *second-nearest-neighbor* vertices x and y are called *prolonged second-nearest-neighbors* if $|x| \neq |y|$ and is denoted by $> x\tilde{,} y <$. The *second-nearest-neighbor* vertices $x, y \in V$ that are not prolonged are called *one-level second-nearest-neighbors* since $|x| = |y|$ and are denoted by $> x\bar{,} y <$. Two vertices $x, y \in V$ are called *third-nearest-neighbor* if $d(x, y) = 3$ (da Silva and Coutinho, 1986).The *third-nearest-neighbor* vertices x and y are called *prolonged third-nearest-neighbors* and is denoted by $< x\tilde{,} y >$ if $x \in W_n$ and $y \in W_(n + 3)$ for some n.

We write $x \prec y$ if the path from x^0 to y goes through x. We call the vertex y a direct successor of x, if $y \succ x$ and x, y are nearest-neighbors. The set of the direct successors of x is denoted by $S(x)$, if $x \in W_n$, then

$$S(x) = \{y_i \in W_(n + 1) | d(x, y_i) = 1, i = 1, 2\}$$

Note that $S(x)$ is a *one-level second-nearest-neighbors*. Below we will consider a semi-infinite Cayley tree of second order, an infinite graph without cycles with 3 edges issuing from each vertex except for x^0 which has only 2 edges. This model is being the generalization of the Vannimenus (1981) model was firstly considered by da Silva and Coutinho (1986).

The Model. For the Ising model with spin values in $\Phi = -1, 1$, the relevant Hamiltonian with competing *first-nearest-neighbor* and *prolonged second-*, and *third-nearest-neighbors* binary interactions has the form

$$H(\sigma) = -J_1 \sum_{<x,y>} \sigma(x)\sigma(y) - J_2 \sum_{>\tilde{x,y}<} \sigma(x)\sigma(y) - J_3 \sum_{<\tilde{x,y}>} \sigma(x)\sigma(y) \tag{1}$$

where the sum in the first term ranges all *first-nearest-neighbors*, the second sum ranges all *prolonged second-nearest-neighbors* and the third sum ranges all *prolonged third-nearest-neighbors*, and the spin variables $\sigma(x)$ assume the values ± 1. Here $J_1, J_2, J_3 \in R$ are coupling constants.

This model is being the generalization of the Vannimenus (1981) model was firstly considered by da Silva and Coutinho (1986).

3. Basic Equations

In this section, we produce the recurrent equations follow the line suggested by Vannimenus (1981), consider the relation of the partition function on V_n on subsets of V_{n-1}. Given the initial conditions on V_2, the recurrence equations indicate how their influence propagates down the tree.For $V_2 = x^0, y^1, y^2, z^1, z^2, z^3, z^4$, where $S(x^0) = y^1, y^2$, $S(y^1) = z^1, z^2$ and $S(y^2) = z^3, z^4$.

Double-trunk Cayley tree. Let $\Gamma^2 = (V, \Lambda)$ be infinite Cayley tree with root x^0. A triple of vertices (x^0, y, z) with $y \in S(x^0)$ and $z \in S(y)$ is called *double-trunk* and respectively the sub tree $\Gamma^2_{(x^0,y,z)} = (V^{(x^0,y,z)}), \Lambda^{(x^0,y,z)})$ is called *double-trunk Cayley tree*, if from root x^0 a single edge $< x^0, y >$ emanates, from vertex y two edges $< x^0, y >, < y, z >$ emanate and from any other vertex $x \in V^((x^0, y, z))$ exactly 3 edges emanate.

It is evident that one can split the semi-infinte Cayley tree Γ^2_+ of second order into four semi-infinite double-trunk Cayley tree $\Gamma^2_{(x^0,y^1,z^1)}, \Gamma^2_{(x^0,y^1,z^2)}$, $\Gamma^2_{(x^0,y^2,z^3)}$ and $\Gamma^2_{(x^0,y^2,z^4)}$. We will consider the partition functions. Let $\Gamma^2_{(x^0,y,z)}$ be a double-trunk Cayley tree and $Z^n_{(x^0,y,z)}(\sigma(x^0), \sigma(y), \sigma(z))$ be a partition function on $V_n^{(x^0,y,z)}$ with fixed $(\sigma(x^0), \sigma(y), \sigma(z))$ on double-trunk (x^0, y, z).

There are a priori 8 different configurations (x^0, y, z) and respectively 8 different partition functions on $Z^n_{(x^0,y,z)}(\sigma(x^0), \sigma(y), \sigma(z))$. Note that for this partition function $Z^n_{(x^0,y,z)}(\sigma(x^0), \sigma(y), \sigma(z))$, we take into account all interactions, except interaction of first-nearest-neighbor $< x^0, y >$ and second-nearest-neighbor $< x^0, z >$. Let

$$\sigma_{V_2} = \begin{pmatrix} \sigma(z^1), \sigma(z^2), \sigma(z^3), \sigma(z^4) \\ \sigma(y^1), \sigma(y^2) \\ \sigma(x^0) \end{pmatrix}$$

be a configuration on the set V_2 and $\Omega(V_2)$ be the set of all configurations on V_2. Assume

$$Z^n(\sigma_{V_2}) = Z^n \begin{pmatrix} \sigma(z^1), \sigma(z^2), \sigma(z^3), \sigma(z^4) \\ \sigma(y^1), \sigma(y^2) \\ \sigma(x^0) \end{pmatrix}$$

be the partition function on V_n with fixed V_2. There are 128 different partition functions $Z^n(\sigma_{V_2})$ to consider and the partition function Z^n in volume V_n can be written as follows

$$Z^n = \sum_{\sigma_{V_2} \in \Omega(V_2)} Z^n(\sigma_{V_2})$$

Assume that

$$a = exp(J_1/(2K_BT)), b = exp(J_2/(K_BT)), c = exp(J_3/(K_BT))$$

Then, one can show that there are only 8 independent variables of $Z^n(\sigma_{V_2})$. We define for convenience the following variables,

$$u_1 = \sqrt[4]{Z^n \begin{pmatrix} +,+,+,+ \\ +,+ \\ + \end{pmatrix}}, u_2 = \sqrt[4]{Z^n \begin{pmatrix} -,-,-,- \\ +,+ \\ + \end{pmatrix}},$$

$$u_3 = \sqrt[4]{Z^n \begin{pmatrix} +,+,+,+ \\ -,- \\ + \end{pmatrix}}, u_4 = \sqrt[4]{Z^n \begin{pmatrix} -,-,-,- \\ -,- \\ + \end{pmatrix}},$$

$$u_5 = \sqrt[4]{Z^n \begin{pmatrix} +,+,+,+ \\ +,+ \\ - \end{pmatrix}}, u_6 = \sqrt[4]{Z^n \begin{pmatrix} -,-,-,- \\ +,+ \\ - \end{pmatrix}},$$

$$u_7 = \sqrt[4]{Z^n \begin{pmatrix} +,+,+,+ \\ -,- \\ - \end{pmatrix}}, u_8 = \sqrt[4]{Z^n \begin{pmatrix} -,-,-,- \\ -,- \\ - \end{pmatrix}},$$

Establishing the following recursive relations is straightforward

$$u_1' = ab(cu_1 + c^{-1}u_2)^2, u_2' = ab^{-1}(cu_3 + c^{-1}u_4)^2$$

$$u_3' = a^{-1}b(cu_5 + c^{-1}u_6)^2, u_4' = a^{-1}b^{-1}(cu_7 + c^{-1}u_8)^2$$

$$u_5' = a^{-1}b^{-1}(c^{-1}u_1 + cu_2)^2, u_6' = a^{-1}b(c^{-1}u_3 + cu_4)^2$$

$$u_7' = ab^{-1}(c^{-1}u_5 + cu_6)^2, u_8' = ab(c^{-1}u_7 + cu_8)^2$$

where prime variables correspond to $Z^{n+1}_{(x^0,y,z)}(\sigma(x^0), \sigma(y), \sigma(z))$. We note that, in the paramagnetic phase (high symmetry phase), $u_1 = u_8, u_2 = u_7, u_3 = u_6$ and $u_4 = u_5$. For discussing the phase diagram, the following choice of reduced variables is convenient,

$$x_1 = (u_2 + u_7)/(u_1 + u_8), x_2 = (u_3 + u_6)/(u_1 + u_8),$$

$$x_3 = (u_4 + u_5)/(u_1 + u_8),$$

$$y_1 = (u_1 - u_8)/(u_1 + u_8), y_2 = (u_2 - u_7)/(u_1 + u_8),$$

$$y_3 = (u_3 - u_6)/(u_1 + u_8), y_4 = (u_4 - u_5)/(u_1 + u_8)$$

Lastly, for the system recurrence equations for the model (1) on a Cayley tree order 2, we have the relations,

$$\begin{aligned}
x_1' &= b^{-2}D^{-1}[(cx_2 + c^{-1}x_3)^2 + (cy_3 + c^{-1}y_4)^2] \\
x_2' &= a^2D^{-1}[(c^{-1}x_2 + cx_3)^2 + (c^{-1}y_3 + cy_4)^2] \\
x_3' &= a^2b^2D^{-1}[(cx_1 + c^{-1})^2 + (c^{-1}y_1 + cy_2)^2] \\
y_1' &= 2D^{-1}[y_1(c^2 + x_1) + y_2(c^{-2}x_1 + 1)] \\
y_2' &= 2D^{-1}[y_3(c^2x_2 + x_3) + y_4(c^{-2}x_3 + x_2)] \\
y_3' &= -2D^{-1}[y_3(c^{-2}x_2 + x_3) + y_4(c^2x_3 + x_2)] \\
y_4' &= -2D^{-1}[y_1(c^{-2} + x_1) + y_2(c^2x_1 + 1)]
\end{aligned} \tag{2}$$

with $D = (c + c^{-1}x_1)^2 + (cy_1 + c^{-1}y_2)^2$. The recurrence equations (2) are different which has produced by da Silva and Coutinho (1986). In the next section, we investigate the behavior of the system in equation (2) by applying numerical simulation.

4. Phase Diagrams

It is convenient to know the broad features of the phase diagram before discussing the different transitions in more detail. This can be achieved numerically in a straightforward fashion. The recursion relations from (2) provide us a numerically exact phase diagram in $((k_BT)/J_1, -J_3/J_1)$ space. $\alpha =$

$(k_BT)/J_1, \beta_2 = -J_2/J_1$ and $\beta_3 = -J_3/J_1$, with $a = exp(\alpha^{-1}), b = exp(\alpha^{-1}\beta_2)$ and $c = exp(-\alpha^{-1}\beta_3)$. Starting from the initial conditions

$$x_1^{(1)} = a^{-4}b^{-2}(b^4c^4 + a^8)/a^8b^4c^4 + 1, x_2^{(1)} = a^{-6}(a^8c^4 + b^4)/(a^8b^4c^4 + 1),$$

$$x_3^{(1)} = a^{-2}b^{-2}(a^8b^4 + c^4)/(a^8b^4c^4 + 1),$$

$$y_1^{(1)} = (a^8b^4c^4 - 1)/(a^8b^4c^4 + 1), y_2^{(1)} = -a^{-4}b^{-2}(a^8 - b^4c^4))/(a^8b^4c^4 + 1),$$

$$y_3^{(1)} = a^{-6}(a^8c^4 - b^4)/(a^8b^4c^4 + 1), y_4^{(1)} = -a^{-2}b^{-2}(c^4 - a^8b^4)/(a^8b^4c^4 + 1)$$

that corresponds to boundary condition $\tilde{\sigma}(n) \equiv \pm 1$, one iterates the recurrence relations in (2) and observes the behavior after a large number of iterations. In the simplest situation, a fixed point $(x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*, y_4^*)$ is reached. That point corresponds to the paramagnetic phase if $y_1^* = y_2^* = y_3^* = y_4^* = 0$ or to the ferromagnetic phase if $y_1^*, y_2^*, y_3^*, y_4^* \neq 0$.

The system may be commensurate with period p, where the case $p = 2$ corresponds to the antiferromagnetic phase and the case $p = 4$ corresponds to the antiphase, where is denoted by $< 2 >$ for compactness. Finally, the system may remain aperiodic. The distinction between a truly aperiodic case and one with a very long period is difficult to achieve numerically. Bellow, we consider periodic phase with period p for $p \geq 12$. We will consider all periodic phases with period $p > 12$ and the aperiodic phase as modulated phase.

The resultant phase diagrams on the plane (β_3, α) for some values β_2 are shown in Figs. 1 - 3. Note that P - paramagnetic, F - ferromagnetic, AF - antiferromagnetic, P3 - phase of period 3, <2> - antiphase, <3> - phase of period 6 and M - modulated phase)

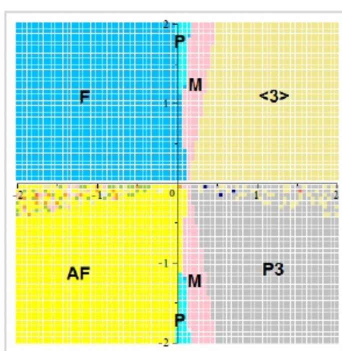


Figure 1: Phase diagram of α vs β_3 for the model with $\beta_2 = 0$.

The phase diagram in Fig. 1 was considered, where the system of equations (2) was iterated for $\beta_2 = 0$. We consider all possible signs of J_3 and J_1, and the plane (β_3, α) is parted into four quadrants. The diagram in the first quadrant consists of four phases: a ferromagnetic phase, a paramagnetic phase, a modulated phase and a phase of period 6. Next, in the second quadrant consists of 2 phases: a ferromagnetic with a small region of paramagnetic. In the fourth quadrant, the diagram consists of four phases: paramagnetic, antiferromagnetic, modulated and phase of period 3. As for the third quadrant, the phase diagram consists of antiferromagnetic phase.

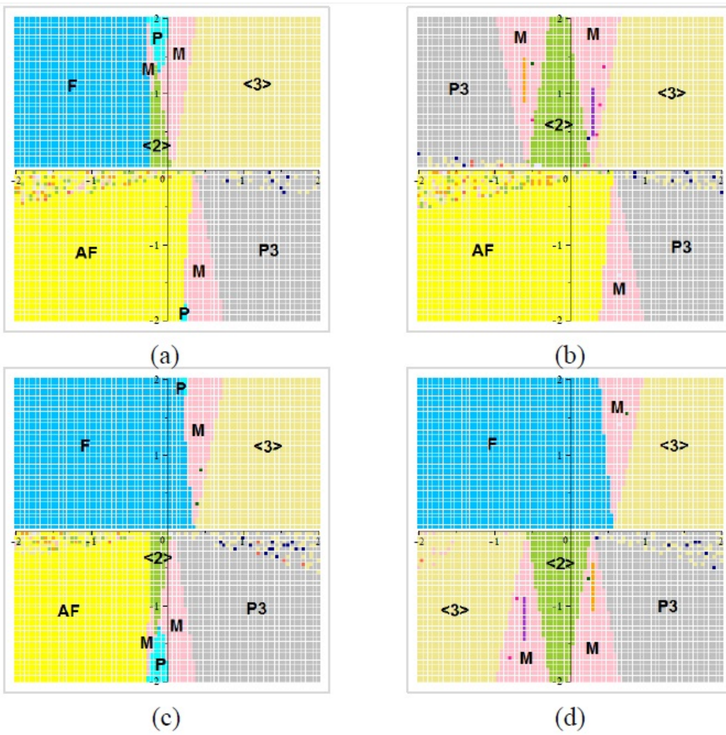
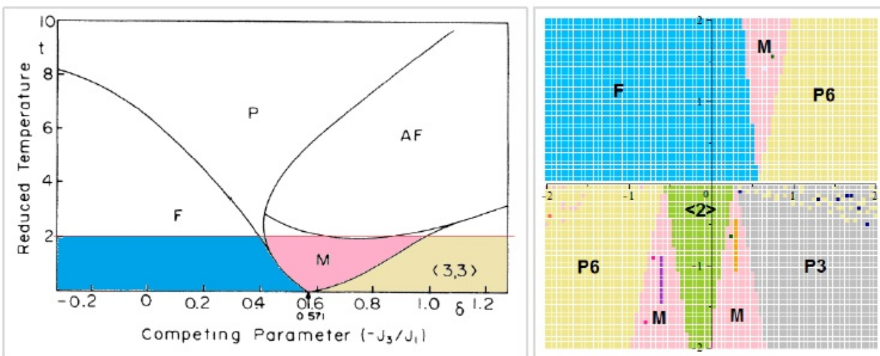


Figure 2: Phase diagram of α vs β_3 for the model with (a) $\beta_2 = -0.5$, (b) $\beta_2 = -1.0$, (c) $\beta_2 = 0.5$ and (d) $\beta_2 = 1.0$.

In Fig. 2, we produce the phase diagrams for the model in (1) with $\beta_2 \neq 0$. We iterate the system of equations (2) and we found that the phase diagrams (Figs. 2(a) and 2(c)) consist of seven phases: ferromagnetic, paramagnetic, antiferromagnetic, phase of period 3, antiphase, phase of period 6 and modulated phases. One can see that the second nearest-neighbor interactions play important role in the result for the phase diagram of the model in (1). We produced the phase diagrams for nonzero β_2, additional antiphase exists in these diagrams. As for the diagram in Fig. 1 with zero β_2, the diagram only consists six phases, without antiphase.

In addition, we found that in Figs. 2(b) and 2(d) a paramagnetic phase does not exists when $\beta_2 = \pm 1.0$ is fixed. As the values $|\beta_2| > \beta_2^c$, for some critical β_2^c, numerically the paramagnetic phase does not occurs in the phase diagrams. In Fig. 3, we compare the phase diagram was produced by Vannimenus approach (Vannimenus, 1981) with the diagram was produced by da Silva and Coutinho (1986). The value of $\beta_2 = 1.0$ has been fixed and one can see that by Vannimenus approach (Fig. 3(b)), the diagram has similar results as founded by da Silva and Coutinho (1986)(Fig. 3(a)). In the first quadrant ($J_1 > 0, J_2 < 0$), both diagrams consist of three phases: ferromagnetic, modulated and phase of period 6, with $0 < \alpha \leq 2$ and $0 < \beta_3 \leq 2$.



(a) (b)

Figure 3: Phase diagram of α vs β_3 for the model with (a) da Silva and Coutinho and (b) Vannimenus approach, $\beta_2 = 1.0$

Thus, from the previous phase diagrams, one can see that an investigation on the behavior of the system in (2) can be performed by applying numerical simulation. In brief, the system undertook a large number of iterations before the detailed could be seen on the (β_3, α) space.

5. Modulated Phases

We will study in detail the set of modulated phases by conducting an investigation on the wavevector by varying the temperature and the Lyapunov exponent, as presented by Vannimenus (1981). A definition of the wavevector that is convenient for numerical purpose is

$$q = \lim_{N \to \infty} (n(N)/2N)$$

where $n(N)$ is the number of times the average magnetization changes sign during N successive iterations Vannimenus (1981). A graph of q versus T is drawn in Fig. 4.

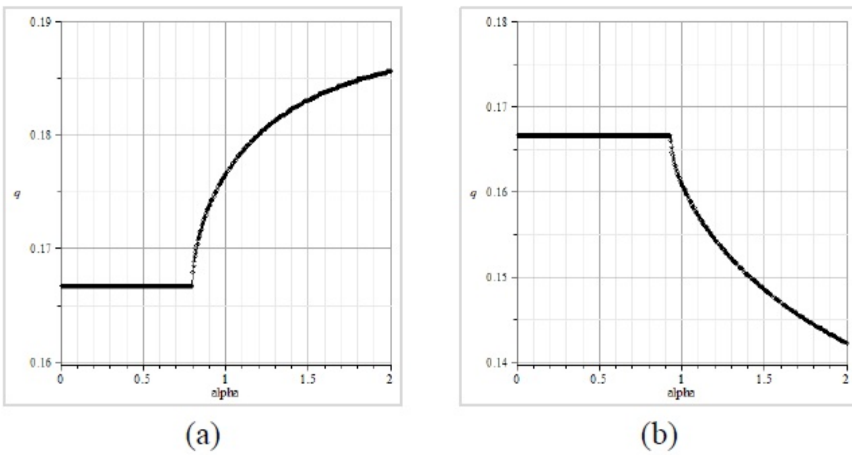


Figure 4: Variation of the wavevector q vs temperature: (a) $\beta_2 = -1.0$,$\beta_3 = 0.45$ and (b) $\beta_2 = 0.5, \beta_3 = 0.55$

In this Fig. 4, one can see that the value of q starts with $q = 1/6$, indicating the phase of period 6 ($< 3 >$) (da Silva and Coutinho, 1986). This phase is very stable, and one can compute the Lyapunov exponent which is negative. We investigate the modulated phase, with the phase indicated by many different values of q. A detailed investigation of $q(T)$ is needed to locate the main locking steps that have to be present according to the general theory. This interval can be very narrow; moreover, the distinction between long periodic cycles and truly aperiodic solutions is difficult to achieve numerically.

The answer to this problem consists in computing the Lyapunov exponent associated with the trajectory of the system, as discussed and presented in Vannimenus (1981). The Lyapunov exponent shows a behavior is which the logarithm of the largest eigenvalue has a simple fixed point. The Lyapunov exponent also tells us whether an infinitesimal perturbation of the initial conditions will have an infinitesimal effect (negative exponent) or will lead to a totally different trajectory (positive exponent).

For the Ising model, the calculation of the Lyapunov exponent can be summarized as follows: the recurrence equations in (2) are linearized around the successive points of the trajectory, yielding linear recurrence equations for the perturbations $(\delta x_1, \delta x_2, \delta x_3, \delta y_1, \delta y_2, \delta y_2, \delta y_3)$ (see discussion in Vannimenus (1981)). In matrix, one has

$$V_{k+1} = (\delta x_1^{'}, \delta x_2^{'}, \delta x_3^{'}, \delta y_1^{'}, \delta y_2^{'}, \delta y_3^{'}, \delta y_4^{'})^T = L_k(\delta x_1, \delta x_2, \delta x_3, \delta y_1, \delta y_2, \delta y_3, \delta y_4)^T$$

where the matrix L_k depends on iteration step. The Lyapunov exponent λ is obtained as

$$\lambda = \lim_{N \to \infty} (1/N \log ||V_N||)$$

where $||V_N||$ denotes the norm of the vector V. Stable limit cycles may exist only for negative exponents.

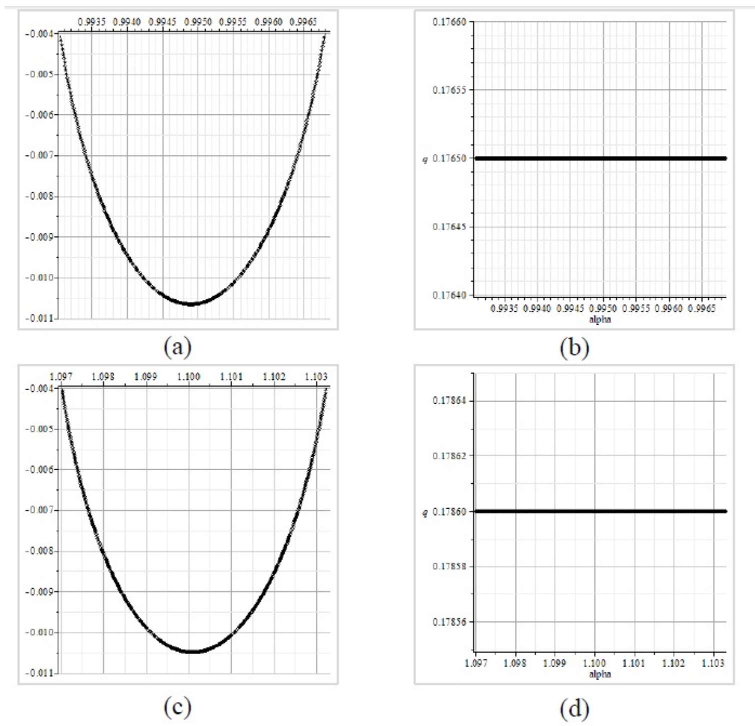
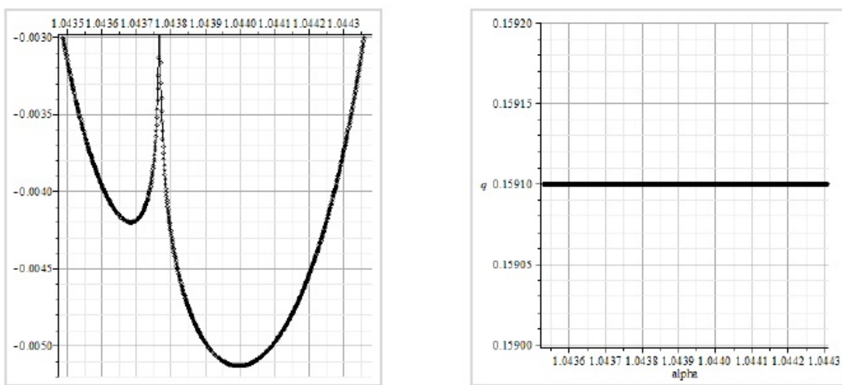


Figure 5: (a) and (c) Variation of the Lyapunov exponent λ for $\beta_2 = -1.0, \beta_3 = 0.45$ in the regions of commensurate steps (b) $q = 3/17$ and (d) $q = 5/28$, enlarged from fig. 4(a).

From Fig. 5, the value of $\beta_2 = -1.0, \beta_3 = 0.45$, was chosen because q is very close to 3/17 and 5/28. The stability intervals for these values of q are rather narrow: from $\alpha \in (0.9930, 0.9968)$ (see Fig. 5(b)) and $\alpha \in (1.0970, 1.1033)$ (see Fig. 5(d)), respectively. We investigated the stability of these phases in the modulated phase, and the calculation of the Lyapunov exponent is presented in Figs. 5(a) and 5(c). Numerically, every region of negative λ is found to coincide with the stability domain of a given cycle.

In Fig. 5(a) shows the results obtained for a cycle of period 17 with $N = 10000$ iteration steps (the initial 1000 steps are discarded for each value of T, as discussed in (Vannimenus, 1981)). Moreover, we show a stability cycle of period 28 for $\beta_2 = -1.0, \beta_3 = 0.45$. In Fig. 6, we consider the case of $\beta_2 = 0.5$ and $\beta_3 = 0.55$, we show other stable limit cycles of the period. The Lyapunov exponent has been calculated for the linearized system in (2) and the value of $\beta_3 = 0.55$ has been fixed (see fig. 6(a)). We found one stable region with $q \approx 7/44$ (see fig. 6(b)), corresponding to a limit cycle of period 44 in the interval $\alpha \in (1.04353, 1.04431)$.



(a) (b)

Figure 6: Variation of the Lyapunov exponent λ for $\beta_2 = 0.5, \beta_3 = 0.55$ in the regions of commensurate steps (b) $q = 7/44$, enlarged from fig. 4(b).

Numerically, every region of negative λ is found to coincide with the stability domain of a given cycle. Finally, all stable limit cycles can be confirmed by checking the trajectory of the system in (2) corresponding to the given interval of α, as presented in Figs. 5 and 6.

6. Conclusions

In this research, we studied the Ising model on a semi-infinite Cayley tree with competing *first-nearest-neighbor* interactions J_1, *second-nearest-neighbor* interactions J_2 and *third-nearest-neighbor* interactions J_3. We constructed the recurrence system of equations, where the system in (2) was produced by Vannimenus approach (Vannimenus (1981)), which different those studied by da Silva and Coutinho (1986). From the consideration above, one can see that both approaches give same result in phase diagram. Moreover, the role of *second-nearest-neighbor* interactions is rather significant. Numerically, for some critical value of β_2, the paramagnetic phase does not occurs in the phase diagrams. In addition, one can observe commensurate phases with some periods p=17,28 and 44 in the set of modulated phases. Lastly, we studied the variation of the wavevector with temperature in a modulated phase by using the Lyapunov exponent associated with the trajectory of the system. We found that stability of a limit cycle of period $p > 12$ can be found in a narrow interval in the modulated phase.

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